

Precise predictions for $B \rightarrow X_s \ell^+ \ell^-$ in the large q^2 region

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The inclusive $B \rightarrow X_s \ell^+ \ell^-$ decay rate in the large q^2 region ($q^2 > m_{\psi'}^2$) receives significant nonperturbative corrections. The resulting uncertainties can be drastically reduced by normalizing the rate to the $B \rightarrow X_u \ell \bar{\nu}$ rate with the same q^2 cut, which allows for much improved tests of short distance physics. We calculate this ratio, including the order $1/m_b^3$ nonperturbative corrections and the analytically known NNLO perturbative corrections. Since in the large q^2 region an inclusive measurement may be feasible via a sum over exclusive states, our results could be useful for measurements at LHCb and possibly for studies of $B \rightarrow X_d \ell^+ \ell^-$.

I. INTRODUCTION

The $b \rightarrow s \ell^+ \ell^-$ process plays an important role in making overconstraining measurements of CKM matrix elements and searching for physics beyond the Standard Model (SM). This decay has been observed both in inclusive $B \rightarrow X_s \ell^+ \ell^-$ [1, 2] and exclusive $B \rightarrow K^{(*)} \ell^+ \ell^-$ [3, 4] transitions. The inclusive $B \rightarrow X_s \ell^+ \ell^-$ decay rate can be calculated in a systematic expansion if one ignores the J/ψ and ψ' resonances. It has thus been advocated to compare calculations and measurements of the (differential) rate for $q^2 < m_{J/\psi}^2$ and $q^2 > m_{\psi'}^2$, which we shall refer to as the small q^2 and large q^2 regions, respectively. Here $q^2 = (p_{\ell^+} + p_{\ell^-})^2$ is the dilepton invariant mass, and in practice the q^2 regions are chosen as $q^2 \lesssim 6 \text{ GeV}^2$ and $q^2 \gtrsim 14 \text{ GeV}^2$.

The measurements in the two regions are complementary, as they have different sensitivities to short distance physics, the main theoretical uncertainties have different origins, and the experimental challenges are also distinct. The most important operators for $B \rightarrow X_s \ell^+ \ell^-$ are

$$\begin{aligned} O_7 &= \frac{e}{16\pi^2} \bar{m}_b(\mu) (\bar{s}_L \sigma_{\mu\nu} b_R) F^{\mu\nu}, \\ O_9 &= \frac{\alpha_{\text{em}}}{4\pi} (\bar{s}_L \gamma_\mu b_L) (\bar{\ell} \gamma^\mu \ell), \\ O_{10} &= \frac{\alpha_{\text{em}}}{4\pi} (\bar{s}_L \gamma_\mu b_L) (\bar{\ell} \gamma^\mu \gamma_5 \ell). \end{aligned} \quad (1)$$

The operator O_7 is important in $B \rightarrow X_s \ell^+ \ell^-$ at small q^2 due to the $1/q^2$ pole from the photon propagator (and it dominates the $B \rightarrow X_s \gamma$ rate). At large q^2 , however, the O_7 contribution is small. Compared to small q^2 , the rate in the large q^2 region has a smaller renormalization scale dependence and m_c dependence [5]. Although the rate is smaller at large q^2 , the experimental efficiency is better [1, 2]. Moreover, requiring large q^2 constrains the X_s to have small invariant mass, m_{X_s} , which suppresses the background from $B \rightarrow X_c \ell^- \bar{\nu} \rightarrow X_s \ell^+ \ell^- \nu \bar{\nu}$. To suppress this background at small q^2 , an upper cut on m_{X_s} is required, complicating the theoretical description due to the dependence of the measured rate on the shape function [6], which is absent at large q^2 [7, 8].

Despite these advantages, the large q^2 region has been considered less favored. The $1/m_b^3$ corrections are not much smaller than the $1/m_b^2$ ones [9], so it is often stated

that the $B \rightarrow X_s \ell^+ \ell^-$ rate in the large q^2 region has a large hadronic uncertainty [7, 9, 10]. The reason is that the operator product expansion becomes an expansion in $\Lambda_{\text{QCD}}/(m_b - \sqrt{q^2})$ [11] instead of Λ_{QCD}/m_b .

Our main point is that this uncertainty can be drastically reduced by comparing measurements and calculations of the ratio

$$\frac{\int_{q_0^2}^{m_B^2} \frac{d\Gamma(B \rightarrow X_s \ell^+ \ell^-)}{dq^2}}{\int_{q_0^2}^{m_B^2} \frac{d\Gamma(B \rightarrow X_u \ell \bar{\nu})}{dq^2}} = \frac{|V_{tb} V_{ts}^*|^2}{|V_{ub}|^2} \frac{\alpha_{\text{em}}^2}{8\pi^2} \mathcal{R}(q_0^2), \quad (2)$$

with the same lower cut $q^2 > q_0^2$ in the $b \rightarrow s$ and $b \rightarrow u$ decays.¹ The nonperturbative corrections related to the dominant O_9 and O_{10} contributions are the same as for the semileptonic rate. Thus, as explained below, nonperturbative effects in the ratio in Eq. (2) are suppressed near maximal q^2 by

$$1 - \frac{(\mathcal{C}_9 + 2\mathcal{C}_7)^2 + \mathcal{C}_{10}^2}{\mathcal{C}_9^2 + \mathcal{C}_{10}^2} \simeq 0.12, \quad (3)$$

which is nearly an order of magnitude. The scheme we use for the Wilson coefficients $\mathcal{C}_{7,9,10}$ [12] will be defined in Sec. III. Their SM values are

$$\mathcal{C}_9 = 4.207, \quad \mathcal{C}_{10} = -4.175, \quad \mathcal{C}_7 = -0.2611. \quad (4)$$

We calculate in this paper the ratio $\mathcal{R}(q_0^2)$, which allows one to translate the measured $B \rightarrow X_s \ell^+ \ell^-$ and $B \rightarrow X_u \ell \bar{\nu}$ rates in the large q^2 region to a precision constraint on the Wilson coefficients times the CKM elements in Eq. (2). The normalization of $\mathcal{R}(q_0^2)$ is chosen such that at lowest order and in the limit $|\mathcal{C}_7/\mathcal{C}_{9,10}| \ll 1$, $\mathcal{R}(q_0^2) = \mathcal{C}_9^2 + \mathcal{C}_{10}^2$. Hereafter we assume the SM and neglect the strange quark and lepton masses.

¹ This was noted as a remote possibility in Ref. [8], but was subsequently forgotten even by those authors. The experimental prospects have improved sufficiently that such a study may be possible in the near future.

II. THE q^2 SPECTRA TO ORDER $1/m_b^3$

The nonperturbative corrections to the q^2 spectrum are calculable in an operator product expansion (OPE) [13]. The first corrections appear at $\mathcal{O}(\Lambda_{\text{QCD}}^2/m_b^2)$ [14, 15]. They are parameterized by two nonperturbative matrix elements, λ_1 and λ_2 . At

$\mathcal{O}(\Lambda_{\text{QCD}}^3/m_b^3)$ there are two new local matrix elements, ρ_1 and ρ_2 , four time-ordered products, \mathcal{T}_{1-4} [16, 17], and process dependent matrix elements of four-quark operators, f_i [18, 19, 20].

The q^2 spectrum up to $\mathcal{O}(\Lambda_{\text{QCD}}^3/m_b^3)$ for $B \rightarrow X_u \ell \bar{\nu}$ is given by [13, 16]

$$\frac{d\Gamma_u}{dq^2} = \frac{G_F^2 |V_{ub}|^2}{192 \pi^3} m_b^3 \left[(1-s)^2 (1+2s) (2+\hat{\lambda}_1) + 3(1-15s^2+10s^3) (\hat{\lambda}_2 - \hat{\rho}_2) + \frac{37+24s+33s^2+10s^3}{3} \hat{\rho}_1 - \frac{16}{(1-s)_+} \hat{\rho}_1 - 8\delta(1-s) (\hat{\rho}_1 + \hat{f}_u) \right], \quad (5)$$

where $s = q^2/m_b^2$, and $1/(1-x)_+ = \lim_{\epsilon \rightarrow 0} [\theta(1-x-\epsilon)/(1-x) + \delta(1-x-\epsilon) \ln \epsilon]$. For $B \rightarrow X_s \ell^+ \ell^-$ [9, 14, 15, 21],

$$\begin{aligned} \frac{d\Gamma_s}{dq^2} = & \frac{\Gamma_0}{2} m_b^3 \left\{ (C_9^2 + C_{10}^2) \left[(1-s)^2 (1+2s) (2+\hat{\lambda}_1) + 3(1-15s^2+10s^3) (\hat{\lambda}_2 - \hat{\rho}_2) + \frac{37+24s+33s^2+10s^3}{3} \hat{\rho}_1 \right] \right. \\ & + 4\mathcal{C}_7\mathcal{C}_9 \left[3(1-s)^2 (2+\hat{\lambda}_1) - 3(5+6s-7s^2) (\hat{\lambda}_2 - \hat{\rho}_2) + (13+14s-3s^2) \hat{\rho}_1 \right] \\ & + \frac{4\mathcal{C}_7^2}{s} \left[(1-s)^2 (2+s) (2+\hat{\lambda}_1) - 3(6+3s-5s^3) (\hat{\lambda}_2 - \hat{\rho}_2) + \frac{-22+33s+24s^2+5s^3}{3} \hat{\rho}_1 \right] \\ & \left. - [(C_9 + 2\mathcal{C}_7)^2 + C_{10}^2] \left[\frac{16}{(1-s)_+} \hat{\rho}_1 + 8\delta(1-s) (\hat{\rho}_1 + \hat{f}_s) \right] \right\}, \quad (6) \end{aligned}$$

where

$$\Gamma_0 = \frac{G_F^2}{48\pi^3} \frac{\alpha_{\text{em}}^2}{16\pi^2} |V_{tb}V_{ts}^*|^2. \quad (7)$$

The nonperturbative parameters in Eqs. (5) and (6) are

$$\begin{aligned} \hat{\lambda}_1 &= \frac{\lambda_1}{m_b^2} + \frac{\mathcal{T}_1 + 3\mathcal{T}_2}{m_b^3}, & \hat{\lambda}_2 &= \frac{\lambda_2}{m_b^2} + \frac{\mathcal{T}_3 + 3\mathcal{T}_4}{3m_b^3}, \\ \hat{\rho}_{1,2} &= \frac{\rho_{1,2}}{m_b^3}, & \hat{f}_{u,s} &= \frac{f_{u,s}}{m_b^3}. \end{aligned} \quad (8)$$

For our purposes, the \mathcal{T}_i can be absorbed into $\lambda_{1,2}$. In the total rate and the q^2 spectrum, λ_1 enters proportional to the b quark decay rate, and the ρ_2 contribution is proportional to λ_2 . Hence, the important nonperturbative parameters for the q^2 spectrum are λ_2 , ρ_1 , and $f_{u,s}$.

The value of λ_2 is known fairly precisely, $\lambda_2 = (m_{B^*}^2 - m_B^2)/4 \simeq 0.12 \text{ GeV}^2$. To estimate ρ_1 , the equations of motion can be used to relate the relevant operator to a four-quark operator. Using the vacuum saturation model, $\rho_1 = (2\pi\alpha_s/9)f_B^2 m_B \simeq (0.4 \text{ GeV})^3$ [17, 22]. The fits to the $B \rightarrow X_c \ell \bar{\nu}$ shape variables are sensitive to ρ_1 and prefer a larger central value [23, 24] with significant uncertainties. We shall use $\rho_1 = (0.1 \pm 0.1) \text{ GeV}^3$.

The four-quark operator contributions, f_u and f_s (sometimes called weak annihilation, though the light quark flavor need not match the flavor of the spectator quark), depend on the final state and on the flavor

of the decaying B meson. They contribute near maximal q^2 , and their contribution has only been derived for the total rate [18, 19, 20, 25] and the lepton energy spectrum [18, 26]. However, the four-quark operators have to be consistently included in the OPE for the fully differential spectrum. This affects the matching for ρ_1 in a nontrivial way at $s = 1$, replacing the singular $\rho_1/(1-s)$ terms present in the earlier literature by the plus distributions in the last lines of Eqs. (5) and (6). Apart from this unambiguous regularization of the singular integrals at $s = 1$, our result in Eq. (6) agrees with Ref. [9].

The values of f_u and f_s are poorly known. They are important, since they are enhanced by a loop factor, $16\pi^2$. In the notation of Ref. [20], $f_u = 2\pi^2 f_B^2 m_B (B_1 - B_2)$, where $B_{1,2}$ are phenomenological ‘‘bag parameters’’. In the vacuum saturation model, $B_1 = B_2 = 1$ in charged B decay and $B_1 = B_2 = 0$ in neutral B decay. This gives a significant suppression with large uncertainty, since the accuracy of the model is poorly known. Because of this sensitivity to cancellations between nonperturbative quantities with comparable magnitudes, the estimates of f_u are uncertain. On general grounds one expects in charged B decay f_u^\pm to be greater in magnitude than f_u^0 in neutral B decay, since in the former case the spectator flavor matches the flavor of the light quark in the four-quark operator. The assumption $|B_1 - B_2| = 0.1$ [20] for charged B decay leads to $|f_u^\pm| = 0.4 \text{ GeV}^3$. This would

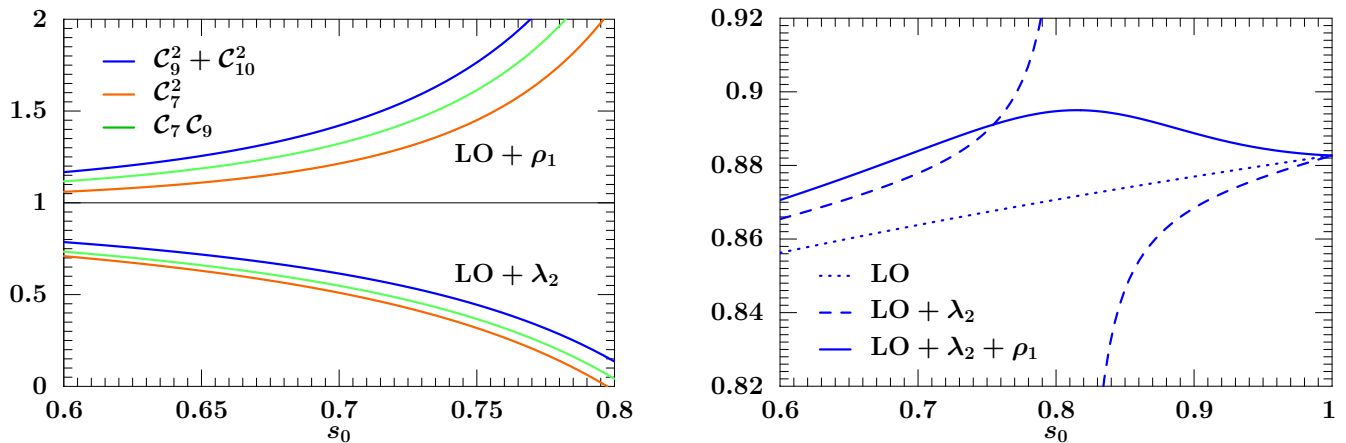


FIG. 1: Left: The impact of the corrections proportional to λ_2 (suppressions) and ρ_1 (enhancements) on the $C_9^2 + C_{10}^2$, $C_7 C_9$, and C_7^2 contributions to the partonic $B \rightarrow X_s \ell^+ \ell^-$ rate integrated over $s_0 < s < 1$. Right: The partonic $B \rightarrow X_s \ell^+ \ell^-$ rate for $s_0 < s < 1$ divided by the $C_9^2 + C_{10}^2$ contribution (the latter is proportional to the $B \rightarrow X_u \ell \bar{\nu}$ rate), at lowest order (dotted), including the λ_2 terms (dashed), and including both λ_2 and ρ_1 corrections (solid). (Note the different scales.)

give a large uncertainty in $\mathcal{R}(q_0^2)$, and we discuss next how it can be reduced.

Flavor $SU(3)$ symmetry implies $f_s^0 \approx f_u^0$. One also expects $f_s^0 \approx f_s^\pm$, though this requires assumptions beyond $SU(3)$ (as there are two singlets in $3 \times \bar{3} \times 3 \times \bar{3}$). Thus, one may use in Eq. (2) the average of the charged and neutral $B \rightarrow X_s \ell^+ \ell^-$ rate and the neutral $B^0 \rightarrow X_u \ell \bar{\nu}$ rate in the presence of the same q^2 cuts.² Separately measuring $B^0 \rightarrow X_u \ell \bar{\nu}$ and $B^\pm \rightarrow X_u \ell \bar{\nu}$ in the large q^2 region is important, even without the additional reasons discussed here, for the determination of $|V_{ub}|$ and to constrain weak annihilation. We anticipate that this separation will be available by the time the $B \rightarrow X_s \ell^+ \ell^-$ rate in the large q^2 region is precisely measured.

To illustrate the slow convergence of the OPE at large q^2 , we show in the left plot in Fig. 1 the effect of the dominant $1/m_b^2$ and $1/m_b^3$ corrections, proportional to λ_2 and ρ_1 , respectively. For each of the $C_9^2 + C_{10}^2$, $C_7 C_9$, and C_7^2 contributions, we plot the free quark decay rate plus either the λ_2 (curves below unity) or the ρ_1 (curves above unity) terms integrated over $s_0 < s < 1$, normalized to the free quark decay rate. The λ_2 term, which is about a -2% correction to the total rate, makes the spectrum negative for $s \gtrsim 0.9$, and the integrated rate negative for $s_0 \gtrsim 0.8$. The terms proportional to ρ_1 give a correction of comparable magnitude and opposite sign (taking $\rho_1 = 0.1 \text{ GeV}^3$). Even at the rather low cut, $s_0 = 0.6$, the correction to the rate proportional to λ_2 is about -21% , while that proportional to ρ_1 is about $+17\%$. Moreover, these estimates do not include the f_i terms discussed above. Thus, even for the rate integrated

over the entire large q^2 region, it is not clear how to assign a robust uncertainty.

The important point is that while these corrections are large, the terms proportional to $C_9^2 + C_{10}^2$ dominate at each order, and they are identical to those that occur in $B \rightarrow X_u \ell \bar{\nu}$. We present a detailed numerical study in the next section. To illustrate the point, the right plot in Fig. 1 shows the partonic $B \rightarrow X_s \ell^+ \ell^-$ rate integrated over $s_0 < s < 1$ divided by the $C_9^2 + C_{10}^2$ contribution, at lowest order (dotted), including the λ_2 terms (dashed), and including both λ_2 and ρ_1 corrections (solid). The blowup of the dashed curve near $s_0 \sim 0.8$ is unphysical and occurs because the rate truncated at $\mathcal{O}(\Lambda_{\text{QCD}}^2/m_b^2)$ becomes negative at slightly different values of s_0 for the $C_9^2 + C_{10}^2$, $C_7 C_9$, and C_7^2 contributions, as can be seen in the left plot. As expected, for $s_0 \rightarrow 1$ the ratio is 0.88 at each order, i.e., unity minus the 0.12 correction in Eq. (3).

III. RESULTS FOR $\mathcal{R}(q_0^2)$

To organize the different short-distance contributions, and combine the nonperturbative and the next-to-next-to-leading order (NNLO) perturbative corrections, we use the scheme introduced in Ref. [12]. The Wilson coefficients $\mathcal{C}_{7,9,10}$ are defined as

$$\begin{aligned} \mathcal{C}_7 &= C_7(\mu) [\bar{m}_b(\mu)/m_b] + \dots, \\ \mathcal{C}_9 &= C_9(\mu) + \dots, \\ \mathcal{C}_{10} &= C_{10}. \end{aligned} \quad (9)$$

where the ellipses denote a minimal set of perturbative corrections, such that $\mathcal{C}_{7,9}$ are μ independent and real in the SM (which is automatic for C_{10}). We use the $1S$ scheme [27], which improves the behavior of the perturbative expansions (for the semileptonic q^2 spectrum both

² For $B \rightarrow X_d \ell^+ \ell^-$, $SU(3)$ implies $f_d^0 \approx f_u^\pm$ and $f_d^\pm \approx f_u^0$. If semileptonic $B_s \rightarrow X_u \ell \bar{\nu}$ decays can be measured in the large q^2 region, that would open up yet another set of applications.

the $\alpha_s^2\beta_0$ [8] and full α_s^2 [28] corrections are known). Consequently, we use $m_b \equiv m_b^{1S}$ everywhere, except for the $\overline{\text{MS}}$ b -quark mass, $\overline{m}_b(\mu)$, in Eq. (9), which is renormalized together with $C_7(\mu)$.

The inclusive decay rate is expressed in terms of the effective Wilson coefficients [12]

$$\begin{aligned} C_7^{\text{incl}}(q^2) &= C_7 + F_7(q^2) + G_7(q^2), \\ C_9^{\text{incl}}(q^2) &= C_9 + F_9(q^2) + G_9(q^2), \end{aligned} \quad (10)$$

which are defined such that all terms on the right-hand side are separately μ independent to the order we are working at. We view the coefficients $C_{7,9,10}$ as the parameters sensitive to physics beyond the SM, which should be extracted from experimental data and compared with their SM predictions. Even if they receive significant new physics contributions, the functions $F_{7,9}(q^2)$ and $G_{7,9}(q^2)$ are likely to be dominated by the SM.

The $F_{7,9}(q^2)$ terms in Eq. (10) contain contributions from the remaining $O_{1-6,8}$ operators in the effective Hamiltonian, for which we employ a partial NNLO treatment. We use the Wilson coefficients at $\mathcal{O}(\alpha_s)$ [29, 30, 31, 32, 33], but only keep the $\mathcal{O}(\alpha_s^0)$ contributions to $F_9(q^2)$ [21, 34]. (Note that $F_7(q^2)$ vanishes at order α_s^0 .) We cannot include the $\mathcal{O}(\alpha_s)$ corrections to $F_{7,9}(q^2)$, because the dominant $O_{1,2}$ contributions are only known analytically in the small q^2 region [35]. They have been evaluated numerically for large q^2 , and lead to a reduced scale dependence and central value [5].

The $G_{7,9}(q^2)$ terms in Eq. (10) contain the calculable $\Lambda_{\text{QCD}}^2/m_c^2$ nonperturbative corrections associated with intermediate $c\bar{c}$ loops [36, 37]. They can be included this way for any differential rate [12], provided $\mathcal{O}(\alpha_s/m_c^2, 1/m_c^4)$ cross terms are neglected. They affect $\mathcal{R}(q_0^2)$ slightly below the 1% level in the large q_0^2 region.

Thus, the q^2 spectra, including up to NNLO perturbative and $1/m_b^3$ nonperturbative corrections, are

$$\begin{aligned} \frac{d\Gamma_u}{dq^2} &= \frac{G_F^2 |V_{ub}|^2}{96 \pi^3} m_b^3 (1-s)^2 [1+2s - \Omega^{99}(s)] + \frac{d\Gamma_u^{1/m}}{dq^2}, \\ \frac{d\Gamma_s}{dq^2} &= \Gamma_0 m_b^3 (1-s)^2 \left\{ (|C_9^{\text{incl}}|^2 + C_{10}^2) [1+2s - \Omega^{99}(s)] \right. \\ &\quad + 4 \text{Re}(C_7^{\text{incl}*} C_9^{\text{incl}}) [3 - \Omega^{79}(s)] \\ &\quad \left. + \frac{4|C_7^{\text{incl}}|^2}{s} [2+s - \Omega^{77}(s)] + \Gamma^{\text{brems}} \right\} + \frac{d\Gamma_s^{1/m}}{dq^2}. \end{aligned} \quad (11)$$

The power suppressed corrections, $d\Gamma_{u,s}^{1/m}/dq^2$, are given by the terms in Eqs. (5) and (6) proportional to $\lambda_{1,2}$, $\rho_{1,2}$, and $f_{u,s}$, appropriately replacing $C_{7,9}$ by $C_{7,9}^{\text{incl}}$ as in Eq. (11). The functions $\Omega^{ij}(s)$,

$$\begin{aligned} \Omega^{99}(s) &= \frac{\alpha_s C_F}{2\pi} [\omega_L^{99}(s) + 2s \omega_T^{99}(s)], \\ \Omega^{77}(s) &= \frac{\alpha_s C_F}{2\pi} [s \omega_L^{77}(s) + 2 \omega_T^{77}(s)], \\ \Omega^{79}(s) &= \frac{\alpha_s C_F}{2\pi} [\omega_L^{79}(s) + 2 \omega_T^{79}(s)], \end{aligned} \quad (12)$$

parameter	central value	uncertainty
μ [GeV]	4.7	$^{+4.7}_{-2.35}$
m_b [GeV]	4.7	± 0.04
m_c [GeV]	1.41	± 0.05
λ_2 [GeV ²]	0.12	± 0.02
ρ_1 [GeV ³]	0.1	± 0.1
f_u^\pm [GeV ³]	0	± 0.4
$f_u^0 - f_s$ [GeV ³]	0	± 0.04
$f_u^0 + f_s$ [GeV ³]	0	± 0.2

TABLE I: Central values and ranges of input parameters. The parameters λ_1 , ρ_2 , \mathcal{T}_i are irrelevant for this work and are set to their central values, $\lambda_1 = -0.27 \text{ GeV}^2$ and $\rho_2 = \mathcal{T}_i = 0$.

contain the $\mathcal{O}(\alpha_s)$ corrections to the matrix elements of the $O_i O_j$ contribution [21, 34, 35, 38, 39] converted to the $1S$ scheme, with $\omega_{L,T}^{ij}(s)$ given in Ref. [12]. We neglect finite bremsstrahlung corrections, Γ^{brems} , associated with $O_{1-6,8}$, because they are negligible at large q^2 [40, 41].

The perturbative uncertainty due to the choice of renormalization scale for α_{em} , which appears in the prefactor of $\mathcal{R}(q_0^2)$ in Eq. (2), can be eliminated by including the relevant higher order electroweak corrections to the $B \rightarrow X_s \ell^+ \ell^-$ rate, which have been studied in Refs. [32, 42] in the small q^2 region.

We present our result for the SM prediction for two different values of q_0^2 , as it is an open question what its most suitable choice is. Using the input values in Table I, with all other inputs as in Ref. [12], we obtain

$$\begin{aligned} \mathcal{R}(14 \text{ GeV}^2) &= C_9^2 + C_{10}^2 + 4.79 C_7^2 + 4.31 C_7 C_9 \\ &\quad + 1.06 C_9 + 2.24 C_7 + 0.95, \\ \mathcal{R}(15 \text{ GeV}^2) &= C_9^2 + C_{10}^2 + 4.27 C_7^2 + 4.10 C_7 C_9 \\ &\quad + 0.97 C_9 + 1.91 C_7 + 0.93. \end{aligned} \quad (13)$$

Using the central values of the \mathcal{C}_i in Eq. (4) and evaluating the uncertainties by varying the parameters within their ranges given in Table I, we find

$$\begin{aligned} \mathcal{R}(14 \text{ GeV}^2) &= 35.55 (1 \pm 0.046_{[f_{u,s}]} \pm 0.012_{[\lambda_2, \rho_1]} \\ &\quad \pm 0.054_{[\mu]} \pm 0.030_{[C_i]}), \\ \mathcal{R}(15 \text{ GeV}^2) &= 35.42 (1 \pm 0.065_{[f_{u,s}]} \pm 0.016_{[\lambda_2, \rho_1]} \\ &\quad \pm 0.051_{[\mu]} \pm 0.030_{[C_i]}). \end{aligned} \quad (14)$$

Eqs. (13) and (14) can be directly compared with experimental measurements to constrain the Wilson coefficients (mainly $C_9^2 + C_{10}^2$) and test the standard model.

The first two uncertainties in Eqs. (14) are due to nonperturbative corrections. They are shown in Fig. 2, where we plot $\mathcal{R}(q_0^2)/\mathcal{R}(14 \text{ GeV}^2)$ as a function of q_0^2 . The most important uncertainty is due to the four-quark operators (weak annihilation), in particular, the difference $f_u - f_s$, which does not cancel in the ratio $\mathcal{R}(q_0^2)$. The green (wide light) region in Fig. 2 corresponds to

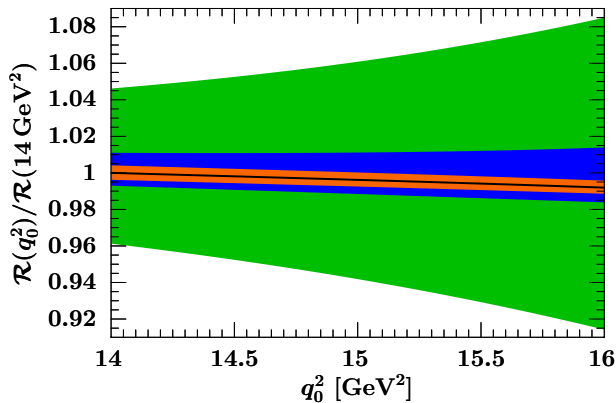


FIG. 2: Nonperturbative uncertainties in $\mathcal{R}(q_0^2)/\mathcal{R}(14 \text{ GeV}^2)$. The black curve is the central value. The green (wide light), blue (dark), and orange (narrow light) shaded bands show the uncertainties from $f_{u,s}$, ρ_1 , and λ_2 , respectively.

allowing $|f_u^0 - f_s| < 0.04 \text{ GeV}^3$ and $|f_u^0 + f_s| < 0.2 \text{ GeV}^3$, which is appropriate if the average of the B^0 and B^\pm rare decay data is compared with the B^0 semileptonic data. Fixing $f_u^0 = f_s^0$, which holds in the $SU(3)$ limit, and varying them together in the range $|f_{u,s}^0| < 0.1 \text{ GeV}^3$, gives a very small uncertainty, below 1%. If $B^0 \rightarrow X_u \ell \bar{\nu}$ is not measured separately from $B^\pm \rightarrow X_u \ell \bar{\nu}$, the uncertainty in $\mathcal{R}(14 \text{ GeV}^2)$ increases to above 20% (taking $|f_u| < 0.2 \text{ GeV}^3$ and $f_s = 0$ as an estimate for the average of the charged and neutral modes).

Even with our conservative range for ρ_1 , the combined uncertainty from ρ_1 and λ_2 is very small, at the 1% level. The allowed variation of λ_2 accounts for $\mathcal{T}_{3,4}$, which it absorbs in Eq. (8), and for ρ_2 , which it is proportional to in Eqs. (5) and (6). The individual uncertainties from ρ_1 and λ_2 are shown in Fig. 2 as the blue (dark) and orange (narrow light) bands, respectively. The total uncertainty from nonperturbative corrections, estimated conservatively, is only about 6%. Without normalizing to the $B \rightarrow X_u \ell \nu$ rate with the same cut, the uncertainty in the $B \rightarrow X_s \ell^+ \ell^-$ rate for $q^2 \geq 14 \text{ GeV}^2$ due to λ_2 , f_s , and ρ_1 would be 4%, 9%, and 21%, respectively.

The remaining two uncertainties in Eqs. (14) are due to the perturbative corrections, and are illustrated in Fig. 3. The variation in μ , shown by the green (wide light) band, is almost entirely due to the $\mathcal{O}(\alpha_s^0)$ pieces in $F_9(q^2)$, because their μ dependence does not cancel in $\mathcal{R}(q_0^2)$. We expect that it will be reduced to below 1% by including the full NNLO result. This is shown by the narrow orange (narrow light) region in Fig. 3, obtained by including the subset of α_s corrections to $F_9(q^2)$ which cancels its leading μ dependence (see Eq. (A5) in Ref. [12]). Finally, the blue (dark) band shows the uncertainty from $\mathcal{C}_{7,9,10}$, which includes their residual μ dependence and dependence on the electroweak matching scale (mainly affecting \mathcal{C}_9), as well as their dependence on the top-quark mass (mainly affecting \mathcal{C}_{10}). The uncertainty in $F_{7,9}(q^2)$ due to the top quark mass is negligible and that

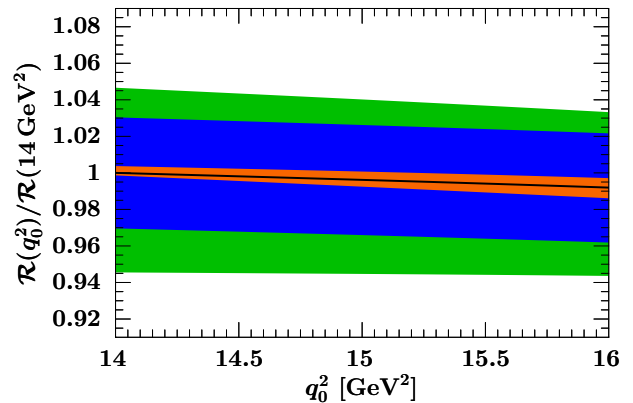


FIG. 3: Perturbative uncertainties in $\mathcal{R}(q_0^2)/\mathcal{R}(14 \text{ GeV}^2)$. The black curve is the central value (as in Fig. 2). The orange (narrow light) and green (wide light) bands show the μ dependence (see text). The blue (dark) band is the uncertainty from $\mathcal{C}_{7,9,10}$.

due to the electroweak matching scale is well below 1%.

The uncertainties due to other input parameters are much smaller than those shown in Eq. (14). The uncertainty from m_c is well below 1%, about the size of that of λ_2 , shown by the orange (narrow light) band in Fig. 2. The m_b dependence is negligible, because it almost completely cancels between the numerator and denominator of $\mathcal{R}(q_0^2)$. This is another significant advantage of considering the ratio $\mathcal{R}(q_0^2)$. Both integrated rates with a cut at $q_0^2 = 14 \text{ GeV}^2$ scale roughly as m_b^{13} , yielding a 11% uncertainty. Normalizing the $B \rightarrow X_s \ell^+ \ell^-$ rate to the total $B \rightarrow X_u \ell \bar{\nu}$ rate (proportional to m_b^5), would still leave about an m_b^8 dependence (and even stronger if normalized to $B \rightarrow X_c \ell \bar{\nu}$), which would give an additional 7% uncertainty compared to our results.

IV. CONCLUDING REMARKS

In this paper we pointed out that the theoretical uncertainty of the $B \rightarrow X_s \ell^+ \ell^-$ rate in the large q^2 region, which is dominated by nonperturbative uncertainties, can be significantly reduced by normalizing the $B \rightarrow X_s \ell^+ \ell^-$ rate to the $B \rightarrow X_u \ell \bar{\nu}$ rate with the same q^2 cut. Fully exploiting this proposal requires the experimental separation of $B^0 \rightarrow X_u \ell \bar{\nu}$ and $B^\pm \rightarrow X_u \ell \bar{\nu}$ in the large q^2 region to eliminate the largest part of the four-quark operator (weak annihilation) contributions.

In much of the theory literature $B \rightarrow X_s \ell^+ \ell^-$ has been normalized to $B \rightarrow X_c \ell \bar{\nu}$ and sometimes to $B \rightarrow X_u \ell \bar{\nu}$. However, to achieve the most reduction of theoretical uncertainties, related to both the matrix elements of the higher dimension operators in the OPE and the value of m_b , the normalization should be done using the $B \rightarrow X_u \ell \bar{\nu}$ rate with the same cuts as in $B \rightarrow X_s \ell^+ \ell^-$. This holds both in the small q^2 region [6], and especially in the large q^2 region studied in this paper.

An uncertainty we have not addressed explicitly is due to higher $c\bar{c}$ resonances, $\psi(3S)$ and above, but their contributions are much smaller than those of the ψ and ψ' and are not expected to introduce a significant uncertainty. Improvements in both theory and experiment will determine what is the optimal choice of the lower limit, q_0^2 , of the large q^2 region to minimize the total uncertainty. If q_0^2 is increased much above $m_{\psi'}^2$, one should get concerned about quark-hadron duality. Even for $q_0^2 = 14 \text{ GeV}^2$ the range of hadronic invariant masses summed over is only $m_{X_s} \leq 1.53 \text{ GeV}$ (and $m_{X_s} \leq 1.41 \text{ GeV}$ for $q_0^2 = 15 \text{ GeV}^2$). Hadronic τ decay data [43] indicate that duality may already be a good approximation at these values.

In the small q^2 region, in addition to the q^2 spectrum and the forward-backward asymmetry, dA_{FB}/dq^2 , one can constrain a third linear combination of the Wilson coefficients by splitting the rate into “transverse” and “longitudinal” components [12]. At large q^2 this is not advantageous, since both are dominated by the $C_9^2 + C_{10}^2$ contribution, and thus yield very similar constraints. Our results can also be applied to the forward-backward asymmetry, A_{FB} , which at large q^2 is mainly sensitive to $C_9 C_{10}$. As noted in Ref. [9], the OPE at large q^2 appears to be significantly better behaved for A_{FB} than for the q^2 spectrum. However, if only the normalized forward-backward asymmetry, $(dA_{\text{FB}}/dq^2)/(d\Gamma/dq^2)$, is measured in the large q^2 region (which may be the least difficult experimentally), it would inherit the large uncertainty of the rate. In this case, one could normalize this measurement to the rate or the normalized A_{FB} in $B \rightarrow X_u \ell \bar{\nu}$ with the same q^2 cut. The latter should be accessible with the large samples of fully reconstructed

B decays used to extract $|V_{ub}|$.

It is not yet known if an inclusive study of $B \rightarrow X_d \ell^+ \ell^-$ can ever be carried out, but it may be less difficult in the large q^2 than in the small q^2 region. At large q^2 a semi-inclusive experimental analysis might become feasible at a super B factory or even at LHCb. The methods discussed in this paper are clearly applicable to this decay as well. Moreover, in the large q^2 region the exclusive rates can be understood model independently using continuum methods [44, 45] or lattice QCD.

To summarize, we showed that it is possible to gain theoretically clean short distance information from the large q^2 region of $B \rightarrow X_s \ell^+ \ell^-$. For this, the experimentally most important input, in addition to precisely measuring $B \rightarrow X_s \ell^+ \ell^-$ at large q^2 , is the measurement of $B^0 \rightarrow X_u \ell \bar{\nu}$ with the same q^2 cut without averaging with $B^\pm \rightarrow X_u \ell \bar{\nu}$. On the theory side, it is desirable to include the full NNLO calculation, $F_{7,9}(q^2)$ at $\mathcal{O}(\alpha_s)$, which will largely reduce the perturbative uncertainties in Fig. 3, and leave us with a prediction for $\mathcal{R}(q_0^2)$ in the 14–15 GeV region with a theoretical uncertainty about the 5% level.

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